

# EPR Paradox , Locality and Completeness of Quantum Theory.

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**Abstract.** The quantum theory (QT) and new stochastic approaches have no deterministic prediction for a single measurement or for a single time -series of events observed for a trapped ion, electron or any other individual physical system. The predictions of QT being of probabilistic character apply to the statistical distribution of the results obtained in various experiments. The Copenhagen interpretation of QT acknowledged the abstract and statistical character of the predictions of QT but at the same time claimed that a state vector  $\Psi$  provided complete description of each individual physical system. The assumption that a state vector was assigned to an individual physical system together with the postulate of its instantaneous reduction in the measurements was shown by Einstein, Podolsky and Rosen to lead to so called EPR paradox for the experiments with entangled pairs of particles. EPR concluded that a state vector could not provide a complete description of individual systems and the question arose whether the probabilistic predictions of QT could be derived from some more fundamental spatio-temporal deterministic description of invisible sub-phenomena by introduction of supplementary parameters. The experimental violation of the Bell inequalities (BI) in the spin polarization correlation experiments (SPCE) which were the implementations of Bohm and EPR gedanken experiments, eliminated so called local and realistic models of the sub-phenomena. Often the violation of BI has been incorrectly interpreted as a proof of the completeness of QT or as the violation of the locality and causality in the micro-world. In this paper we show that local and realistic models overlooked the fact that an experimental outcome is only the information about a particular system-system or system -instrument interaction. Quantum phenomena are described in terms of probabilities. It is well known that the probability distribution is not an attribute of a dice but it is a characteristic of a whole random experiment : " rolling a dice". Therefore quantum probabilities are "contextual" because they describe the lack of knowledge of the outcomes of experiments in contrast to the lack of knowledge of some attributive properties of individual physical systems. We recall that the existence of long range correlations between two random variables X and Y is not a proof of any causal relation between these variables. Moreover any probabilistic model used to described a random experiment is consistent

only with a specific protocol telling how the random experiment has to be performed. The probabilistic model used to prove BI implied a protocol completely inappropriate and impossible to implement for SPCE . Therefore we conclude that the important question whether QT is predictably complete is still open and we show how the unconventional analysis of the existing data could help to answer it. The correct understanding of statistical and contextual character of QT is essential for the research in the domain of quantum information and quantum computing.

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## 0.1 Introduction

It has been shown many years ago [1-6] that the main prerequisites used to prove the Bell inequalities (BI) [7,8], or Clauser,Horn ,Shimony,Holt (CHSH) inequalities [9] such as the use of a common probability space, joint probability distributions for non-commuting observables etc. were inappropriate for the description of the spin polarization correlation experiments (SPCE). Therefore the violation of BI-CHSH in these experiments [10,11] gave neither information about the nonlocality of QT nor about its completeness.

In spite of this, testing of BI-CHSH continued and several loopholes were indicated which could explain the apparent, but not existing, violation of BI-CHSH by the imperfection of the experimental set ups [12]. Precise new experiments [13,14] permitted to close several loopholes, confirmed QT predictions for the correlation functions and even allowed to detect strange anomalies in various detection rates reported recently by Adenier and Khrennikov [15]. Instead of looking for new loopholes it would be more interesting, in our opinion, to use the existing data in order to find anomalies similar to those reported in [15] or to search for some fine structure in the data with help of various purity tests proposed by us several years ago [4,16,17]. Only the discovery of the fine structure in the data, not accounted for by QT , would provide a decisive proof that the statistical description provided by QT of these data is incomplete ending the debate started by EPR [18] over 75 years ago.

The limitations and inapplicability of BI , CHSH and GHZ inequalities [19] to SPCE have been demonstrated now in numerous publications e.g. [20-27,46-50]. Several local models [2,3,28,29,41] were able to reproduce QT predictions. BI-CHSH are also violated in macroscopic experiments discussed by Aerts [30,31] and in computer experiments of Accardi and Regoli [21]. The strong arguments in favor of statistical and contextual character of spin projections were given by Allahverdyan, Balian and Newenhuizen [32]. Instead of rejoicing that there was no contradiction between QT and locality [25] many members of physics community continue to marvel at the picture of two perfect random dices giving completely correlated outcomes etc.

One may wonder why it is so ? One reason could be that they do not understand the implications of the use of a common probability space and joint probability distributions for the protocol of a random experiment . Perhaps the authors criticizing BI were using too technical or/and too condensed language e.g.:" The main hypothesis needed to prove Bell-type inequalities is the assumption that the probabilities estimated in various SPCE can be calculated from one sample space ( probability space) by conditionalization." [5] .

The other reason could be that after Harry Potter and other science fiction stories everything seems to be possible and magic explanations for the correlations are much more attractive than those based on a common cause or a common sense.

Similar reasons may explain perhaps the fact known for years and explained recently in detail by Ballentine [34] that : "Once acquired, the habit of considering an individual particle to have its own wave function is hard to break.... though it has been demonstrated strictly incorrect,".

Fortunately we are at the fourth Vaxjö conference and in my opinion slowly a consensus is starting to build up around statistical contextual interpretation (SCI) of QT [20,27,33-35]. According to this interpretation the information gathered in the measurements in all different experimental contexts provides the only reliable contextual information about a "state" of the identically prepared physical systems whose properties are measured.

It is not an easy process because during the conference several discussions showed that people are attributing different meaning to the words such as : probability, contextual, observables, measurement, photon and local realism and they have their own mental images of invisible sub-phenomena underlying every physical observable phenomenon or experiment.

In this paper we will give additional arguments in favor of SCI. The paper is organized as follows. In the section 2 we define the meaning which we attach to the terms and notions such as: phenomena, sub-phenomena, physical reality, filters, contextual, observables, probabilities etc. In the section 3 we recall shortly the EPR paradox and the explanation given by SCI. In the section 4 we explain why according to SCI there is no strict anti-correlations of "measured" spin projections for each "individual EPR pair" in the singlet state. In the section 5 we criticize in historical perspective various proofs of BI-CHCH including the most recent one given by Larsson and Gill [36] and we explain why the prerequisites used in these proofs are not valid in SPCE. In the section 6 we discuss how the predictable completeness of QT could be tested.

## 0.2 IMPORTANT NOTIONS

The main assumption in physics is that there exists a material world (physical reality) in which we can observe various phenomena and with which we may interact by performing various repeatable experiments. Another assumption is the existence of physical laws which are responsible for the richness of phenomena and for the regularities observed in our experiments. We have no doubt that the Moon exists when we do not look at it but of course when we look at it

we can perceive it in different colors, in different phases etc. We do not have intuitive images of electrons and photons but we have the abstract mathematical model given by QT able to describe quantitatively various phenomena they are participating in. The importance of the interplay of ontic and epistemic realities in QT was recently discussed in some detail by Atmospacher and Primas [37] and Emch [38].

The mathematical models are always of limited validity and apply to particular phenomena. To describe the motion of the Moon around the Earth we use a model of a material point following well defined trajectory obtained by solving Newton's differential equations of motion. To explain its phases the Moon is modeled as a sphere. To describe the details of the formation of a crater on the Moon, when a meteorite hits it, we need much more detailed and complicated mathematical model. This is why we will probably discover one day that at very short distances the extendedness of the hadrons and of other elementary particles plays more important role [39] than in the standard model we are using today.

Therefore the statement that a given theory for example QT provides the most complete description of the individual physical systems lacks humility since as Bohr said [40]: "The main point to realize is that the knowledge presents itself within a conceptual framework adapted to account for previous experience and that any such frame may prove too narrow to comprehend new experiences."

### 0.2.1 Phenomena and Sub-Phenomena

Phenomena produce observable effects. Sub-phenomena are invisible. For example we can see a track left by a single charged high energy elementary particle on the picture from the bubble chamber. To describe its trajectory we can use a model of a point-like particle moving according to the laws of classical physics but of course there is an underlying microscopic invisible sub-phenomenon leading to the track formation and if we wanted to explain it in detail we should go beyond the classical electrodynamics. If this high energy elementary particle collides with the proton from a hydrogen atom in a bubble chamber then we see many out-going tracks from the collision point what is a new phenomenon : a creation of several new particles during the collision. To describe this phenomenon we have to go beyond the quantum electrodynamics. We have to prepare in an accelerator a collimated beam of "identical elementary particles" and to observe several collision events in the bubble chamber. The only reproducible regularities, besides the conservation laws, we assumed to be valid, are of statistical nature and can be predicted by some abstract mathematical model providing the probabilities for different possible outcomes of the collision. The intuitive description of invisible sub-phenomena taking place during the collision is not provided.

The phenomena described by QT are all of this nature. Any experimental set-up can be divided into three parts: a source preparing the ensemble of "identical" physical systems, an interaction/filtering part and the detectors/counters part which produces time-series of observable events : clicks on

various detectors, dots on a screen, tracks in a bubble chamber etc. One may have an impression that nowadays we are able to perform the experiments on the individual physical systems such as an electron or an ion in some trap. However in order to find any statistical regularity in the data from these experiments we have to reset initial conditions in the trap and repeat these experiments several times obtaining again the ensemble of measurements performed on the "identical" physical systems. For the extensive discussion of general experimental set-ups see [42]. QT gives the probabilistic predictions for the distribution of outcomes obtained in various phenomena without providing intuitive models of invisible sub-phenomena. One encounters paradoxes only if incorrect models are used to describe these sub-phenomena. A source of light is not a gun and photons are not small bullets etc.

### 0.2.2 Attributes and Contextual Properties

An attributive property is a constant property of an object which can be observed and measured at any time and which is not modified by the measurement e.g.: inertial mass, electric charge etc. A contextual property is a property revealed only in specific experiments and characterizes the interaction of the object with the measuring apparatus. Let us mention Accardi's chameleon which is green on a leaf and brown on a bark of a tree [22,25].

In classical physics we assume that measurements of various attributive properties possessed by an individual physical system are compatible what means that they can be measured simultaneously or in any sequence giving always the same results. In quantum physics the contextual properties are known after the measurements e.g. as a click on a detector placed behind a polarization filter. The measurements of incompatible properties cannot be performed simultaneously and the measurement of one of them destroys the information about the other one. Various sequences of these measurements lead to various probability distributions of the outcomes [42]. The measurements of attributive properties are called by Accardi [25] passive dynamical systems and those of contextual properties active dynamical systems.

In QT contextual properties of individual systems are of statistical character because they may be only deduced from the properties of pure ensembles they are members of [41] "...a value of a physical observable, here a spin projection, associated with a pure quantum ensemble and in this way with an individual physical system, being its member, is not an attribute of the system revealed by a measuring apparatus; it turns out to be a characteristic of this ensemble created by its interaction with the measuring device. In other words the QM is a contextual theory in which the values of the observables assigned to a physical system have only meaning in a context of a particular physical experiment".

Another argument in favor of SCI of QT comes from probability theory. The probabilities are only the "properties" of random experiments [5]: "talking about the probabilities we should always indicate the random experiment needed to estimate their values". QT provides the algorithms to find various probabilities therefore it is a contextual theory. The contextuality in this sense is

the fully objective property of the Nature. Even if nobody observes the collisions of high energy protons they are described by the same probability distributions of the possible outcomes no matter where it happens.

The probabilities found in QT do not describe one particular random experiment but a whole class of equivalent random experiments which are assumed to be repeatable as many times as needed.

### 0.2.3 Probabilities, Correlations and Causality

It is not obvious how to define the probability, what is the randomness etc. These important topics were discussed recently in detail by Khrennikov in his stimulating book [20].

We illustrate here these difficulties by two simple examples, the more detailed discussion may be found in [20,22,42]

1) Let us consider a random experiment which can give only two outcomes: 1 or -1. We repeat this experiment  $2n$  times and we obtain a time series of the results: 1,-1,1,-1,...,1,-1. By increasing the value of  $n$  the relative frequency of getting 1 can approach  $1/2$  as close as we wish suggesting that the probability of getting 1 in each experiment is equal to  $1/2$ . Of course it is incorrect because if we analyze the time series of outcomes in detail we see that we have a succession of the couples of two deterministic experiments or one deterministic experiment keeping memory of the previous result which is called a periodic two dimensional Markov Chain.

2) Let us toss now a fair coin assigning 1 for "head" and -1 for "tail". We get again a time series: 1,1,-1,-1,1,-1..with relative frequencies which tend to  $1/2$ . There is no apparent structure in this series so the hypothesis of the independent and identical repetitions of the same experiment seems to be satisfied and we say that a complete description of this experiment is provided by a single number :a probability of getting 1 which is equal to  $1/2$ . Of course we believe that if we knew all parameters describing the sub-phenomena of tossing experiment we could predict each individual result using the laws of the classical physics. The randomness is here only due to the lack of control of these parameters.

The statisticians and probabilists invented many tests in order to test the randomness and independence in such time series but conclusions from any statistical study is valid only on a given level of confidence. Moreover the series formed by the consecutive decimals of the number  $\pi$  passed all the tests of the randomness in spite of the fact that any consecutive decimal is strictly determined. Without any completely conclusive measure of randomness we have to limit ourselves for any practical applications to generators of pseudo-random numbers which passed the known tests of the randomness.

One of the postulates of Copenhagen interpretation was that in a measurement process a measured value of a physical observable is chosen among all possible values of this observable with a given probability and in a completely random way. It was also believed that this indeterministic behavior of quantum ensembles could not be explained by the lack of control of some hidden variables describing deterministic interactions of individual members with measuring de-

vices. This intrinsically indeterministic behavior of individual quantum "particles" was believed to provide a new standard of the randomness which could serve to produce the unbreakable keys in quantum cryptography.

The fact that the strong correlations created by the source in SPCE survive the filtration and measurement processes is a strong argument against purely random behavior of individual systems during these processes. The individual systems have to carry a memory of their preparation coded in some parameters and a measuring device described by its own uncontrollable microscopic parameters has to act in some deterministic way to produce an observable outcome without destroying completely the memory how the systems were prepared at the source. It is well known that a strong correlation between two random variables has nothing to do with a causal relation between these variables. For example the average price of oil in a given year is correlated with the average salary of Anglican priests in the same year due to the common cause which is inflation. The existence of strong correlations between non-interacting physical systems which interacted in the past was analyzed for the first time by EPR [18] and led them to conclude that the description of these phenomena provided by QT is incomplete. In the next section we discuss shortly their paper.

### 0.3 EPR-PARADOX

EPR consider two systems I and II which are permitted to interact from  $t=0$  to  $t=T$  and which evolve freely and independently afterwards. The state of I or II for  $t \geq T$  can be found only by the reduction of wave packet. Let  $a_1, a_2, a_3, \dots$  be the eigenvalues of some physical observable A to be measured on the system I and  $u_1, u_2, u_3, \dots$  a complete set of corresponding orthogonal eigenfunctions. At the moment of measurement  $T_1 \geq T$  of the observable A on the system I the wave function  $\Psi(x_1, x_2)$  of the system I+II is given by

$$\Psi(x_1, x_2) = \sum_n \psi_n(x_2) u_n(x_1) \quad ((1))$$

If the measurement of A gives  $a_k$  then the wave function is reduced to  $c_k \psi_k(x_2) u_k(x_1)$  where  $\psi_k(x_2)$  is, up to normalization constant, the wave function of the system II immediately after the measurement of A on the system I has been completed and the result  $a_k$  known.

If instead of A we decided at  $t=T_1$  to measure on I another non-commuting observable B with the eigenvalues  $b_1, b_2, b_3$  and a complete set of orthogonal eigenfunctions  $v_1, v_2, v_3, \dots$  then instead of the formula (1) we would have

$$\Psi(x_1, x_2) = \sum_s \varphi_s(x_2) v_s(x_1) \quad ((2))$$

If  $b_r$  was obtained then the wave of the system II immediately after the measurement of B on the system I would have been, up to normalization constant,  $\varphi_r(x_2)$ .

In their paper EPR conclude : " Thus it is possible to assign two different wave functions ( in our example  $\psi_k(x_2)$  and  $\varphi_s(x_2)$  ) to the same reality (the

second system after the interaction with the first” . In each case the functions are assigned with certainty and without disturbing the system II. If one assumes that the wave functions are in one to one correspondence with the states of individual physical systems one obtains a contradiction called EPR Paradox.

Of course according to SCI there is no paradox because the wave functions  $\psi_k(x_2)$  and  $\varphi_s(x_2)$  describe only different sub-ensembles of the ensemble of the particles II. The eigenvalue expansions (1) and (2) being mathematical identity describe different incompatible experiments. They imply the existence of the long-range correlations between the measurements performed on the non-interacting separated physical systems . The state  $\Psi(x_1, x_2)$  is not factorized and is an example of the entangled state so popular nowadays. The discussion following the EPR paper was recently reviewed in detail in [27]. Some arguments of EPR were rejected but nobody was able to prove that QT provided the complete description of individual physical systems.

#### 0.4 SINGLET STATE

The most studied example of the entangled state is a singlet spin state. Spin version of the EPR experiment was proposed by Bohm [43]. Using SCI we analyze here only the predictions of QT for SPCE. The detailed discussion of EPR-B paradox in a spirit of SCI may be found in [27].

The singlet spin state vector for the a system of two particles has the form:

$$\Psi_0 = (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \sqrt{1/2} \quad ((3))$$

where the single particle vectors  $|+\rangle$  and  $|-\rangle$  denote ”spin up” and ”spin down ” with respect to some common coordinate system.

If we ”measure” the spin of the particle #1 along the unit direction vector  $\mathbf{a}$  and the spin of the particle #2 along the unit direction vector  $\mathbf{b}$  , the results will be correlated and for the singlet state the correlations are described by the correlation function:

$$E(\mathbf{a}, \mathbf{b}) = \langle \Psi_0 | \sigma_{\mathbf{a}} \otimes \sigma_{\mathbf{b}} | \Psi_0 \rangle = -\cos \theta_{\mathbf{ab}} \quad ((4))$$

where  $\sigma_{\mathbf{a}} = \boldsymbol{\sigma} \bullet \mathbf{a}$  and  $\sigma_{\mathbf{b}} = \boldsymbol{\sigma} \bullet \mathbf{b}$  denote the components of the Pauli spin operator in the directions of the unit vector  $\mathbf{a}$  and  $\mathbf{b}$  respectively and  $\theta_{\mathbf{ab}}$  is the angle between the directions  $\mathbf{a}$  and  $\mathbf{b}$ . Since  $E(\mathbf{a}, \mathbf{b}) = -1$  for  $\theta_{\mathbf{ab}} = 0$  it was concluded that the results of the spin projection measurements for each individual couple of particles are strictly anti-correlated. It was pointed out in [41] that this conclusion is unjustified since according to SCI the state vector  $\Psi_0$  allows only to find statistical distribution of outcomes without giving a deterministic prediction for any individual outcome. The reason is that sharp directions and angles do not exist in the Nature. Fuzzy measurements in QT have been studied for years and many important results have been obtained [44,45].

Each spin polarization correlation experiment (A,B) is defined by two macroscopic orientation vectors  $\mathbf{A}$  and  $\mathbf{B}$  being some average orientation vectors of



the analyzers [41,26,27]. More precisely the analyzer  $A$  is defined by a probability distribution  $d\rho_A(\mathbf{a})$ , where  $\mathbf{a}$  are microscopic direction vectors  $\mathbf{a} \in O_A$  and  $O_A = \{\mathbf{a} \in S^{(2)}; |1 - \mathbf{a} \cdot \mathbf{A}| \leq \varepsilon_A\}$ . Similarly the analyzer  $B$  is defined by its probability distribution  $d\rho_B(\mathbf{b})$ . Therefore even if the detectors and filters were perfect, no detection loophole, the idealized QT prediction for the correlation function  $E(\mathbf{A}, \mathbf{B})$  would be given not by the formula (4) but by a smeared formula:

$$E(\mathbf{A}, \mathbf{B}) = \int_{O_A} \int_{O_B} -\cos \theta_{\mathbf{ab}} d\rho_A(\mathbf{a}) d\rho_B(\mathbf{b}) \quad ((5))$$

The quantitative effect of smearing of  $\cos \theta_{\mathbf{ab}}$  in the formula (5) can be very small but  $E(\mathbf{A}, \mathbf{A}) \neq -1$  and there are no strict anti-correlations between measured polarization projections. Of course in SPCE the formulas (4) and (5) have to include additional factors to account for the efficiencies of detectors, various transmission coefficients etc. The formula (5) and similar formulas for joint probabilities of detection [41,26,27,32] confirm only the fundamental contextuality of QT due to which a spin projection on a given axis is not a predetermined attribute of an individual physical system recorded by a measuring device but it is created in the interaction of the system with this device. As we already told the correlations between far away measurements suggest the existence of supplementary parameters keeping the memory of the preparation stage and describing the invisible sub-phenomena during the measurement process..

## 0.5 BELL INEQUALITIES

Let us describe a typical SPCE in the language of observed phenomena.

A pulse from a laser hitting a non-linear crystal produces two correlated physical fields propagating with constant velocities towards the far away detectors. Each of these fields has a property that it produces clicks when hitting the photon- detector. We place two polarization analyzers  $A$  and  $B$  in front of the detectors on both sides and after interaction of the fields with the analyzers we obtain two correlated time- series of clicks on the detectors. Each analyzer is characterized by its macroscopic direction vectors, which may be changed at any time. By changing the direction vectors we have various coincidence experiments labeled by  $(A, B)$  where  $\mathbf{A}$  and  $\mathbf{B}$  are the macroscopic direction vectors for the analyzers  $A$  and  $B$  respectively.

In QT the crystal is described as a source of couples of photons in a spin singlet state and the ensemble of these couples is described approximately by the state vector (3). It is well known that the individual photons are neither localizable nor visible and they do not behave as point-like particles following some classical trajectories. Nevertheless the mental picture of correlated photon pairs travelling across the experimental set-up and carrying their own unknown spins (intrinsic magnetic moments) whose projections on any direction are predetermined by a source and recognized by the polarization analyzers

is commonly used in the discussions of SPCE. Knowing that such description of the sub-phenomena is inaccurate Bell tried to formulate the most abstract probabilistic local hidden variable model in order to explain the spin polarization correlations in a singlet state predicted by QT. As we told above in the experiment (A,B) two time -series of outcomes are produced with each outcome being 1 or -1. The main assumptions in so called local realistic hidden variable model (LRHV) proposed by Bell [7] are:

1. Individual outcomes are produced locally by corresponding analyzers A and B.

2. There are some uncontrollable hidden variables  $\lambda \in \Lambda$  determining the value of individual outcomes. In the experiment (A,B) the outcomes are obtained as values of some bi-valued functions on  $\Lambda$  such that  $A(\lambda, \mathbf{a}) = \pm 1$  and  $B(\lambda, \mathbf{b}) = \pm 1$  respectively where  $\mathbf{a}$  and  $\mathbf{b}$  denote the settings of the analyzers (we keep here for purpose the original Bell notation  $\mathbf{a}$  and  $\mathbf{b}$  instead of  $\mathbf{A}$  and  $\mathbf{B}$  used above).

3 .The probability space  $\Lambda$  and the probability distribution  $\rho(\lambda)$  do not depend on  $\mathbf{a}$  and  $\mathbf{b}$ .

Since the probability distribution of hidden variables is prepared at the source far away from the detectors Bell wrongly believed that the assumption 3 is another consequence of the locality. The assumptions 1-3 allow to write the correlation function  $E(\mathbf{a}, \mathbf{b})$  as:

$$E(\mathbf{a}, \mathbf{b}) = \int_{\Lambda} A(\lambda, \mathbf{a}) B(\lambda, \mathbf{b}) \rho(\lambda) d\lambda \quad ((6))$$

Using the formula (6) for any couple of directions of analyzers, BI and CHSH inequalities below can be proven.

$$|E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{b}')| + |E(\mathbf{a}', \mathbf{b}') + E(\mathbf{a}', \mathbf{b})| \leq 2 \quad ((7))$$

In 1976 we met John Bell in Geneva, and we left him few handwritten pages with our comments concerning the limitations of his proof. In particular we pointed out that if the hidden variables  $\lambda$  describing each pair of "particles" were couples of bi-valued, strictly correlated, spin functions  $S_1$  and  $S_2$  on a sphere such that measured outcomes for each pair were the values of  $S_1(\mathbf{a})$  and  $S_2(\mathbf{b})$  then one could not use the integration over the set of all of these functions as it is done in the formula (6). Moreover we indicated that in this case one should try to prove BI by using the estimates of the correlation functions : the empirical averages obtained by averaging the sums of the products  $S_1(\mathbf{a})S_2(\mathbf{b})$  over all pairs in long runs of the corresponding experiments. If  $E(\mathbf{a}, \mathbf{b})$  is replaced by its estimate the proof of (7) may never be rigorous because the error bars have to be included and one has also to assume that the sets of spin functions describing the couples in the runs from different experiments are exactly the same what is highly improbable due to the richness of the uncountable set of

spin functions on a sphere. These ideas in their final more mature form were only published later in a series of papers [4,5,16,41].

In the meantime Pitovsky [2,3] constructed the spin functions on the sphere for which the integral (6) could not be defined and proposed a local hidden variable model based on these functions able to reproduce the predictions of QT and violating BI. Aerts[6,30] inspired by Accardi's paper[1] showed that BI can also be violated in macroscopic experiments. De Baere[46,47] pointed out that BI might be violated due to the non-reproducibility of a set of hidden variables.

The Pitovsky model was difficult to understand. We simplified it and rendered fully contextual in [41]. The spin functions were strictly correlated but due to the smearing over the microscopic directions there was no strict anti-correlations.

In [5] we gave several arguments why BI could not be proven rigorously using the empirical averages and we showed that the use of the unique probability space implied the experimental protocol which was incompatible with SPCE. In conclusion we wrote: "The various SPCE cannot be replaced by one random experiment of the type discussed above and in our opinion this is the reason why the Bell inequalities do not hold. The various probabilities appearing in their proofs are counter-factual and have nothing to do with the measured ones"

Let us clarify here about which one random experiment we were talking. The  $k$ -variate random variable  $X=(X_1, \dots, X_k)$  and joint probability distribution on a unique probability space  $\Lambda$  were invented in order to describe the following random experiment: we take a large random sample of members from some population we measure the values of  $X_i$   $i = 1, \dots, k$  on each member in a sample obtaining an individual outcome for the experiment as a set of  $k$  numbers  $(x_1, \dots, x_k)$ . The empirical joint distribution for the frequencies of these outcomes gives the information about the joint probability distribution characterizing the whole population. From this joint probability distribution one may obtain by conditionalization the marginal probability distributions for any single random variable  $X_i$  or for any group of them. This is exactly the protocol of the random experiment implied by the formula (6): pick up a pair described by  $\lambda$ , measure the spin projections for this pair in all directions etc. what is of course impossible. We thought that the arguments presented against BI were convincing enough to stop further speculations concerning their violation but we were wrong. Probably some papers were simply unknown or not understood. With growing interest in quantum information and speculations about faster than light communications in EPR experiments it was necessary to provide an up-to-date refutation of BI and of nonlocality of QT. It was done e.g. by Accardi et al. [22,25], Khrennikov[20], Hess and Phillip[24], Kracklauer[28] and by myself.[22,26,27]. We already explained why the formula (6) did not apply to SPCE.

The correct formula which may be used to describe locally the sub-phenomena in any particular experiment SPCE for the couple of analyzers (A,B) is:

$$E(\mathbf{A}, \mathbf{B}) = \int_{\Lambda_{AB}} A(\lambda_1, \mathbf{a}) B(\lambda_2, \mathbf{b}) \rho(\lambda_1, \lambda_2) d\rho_A(\mathbf{a}) d\rho_B(\mathbf{b}) d\lambda \quad ((8))$$

where  $\Lambda_{AB} = \Lambda_1 \times \Lambda_2 \times \Lambda_A \times \Lambda_B$  with  $(\lambda_1, \lambda_2, \mathbf{a}, \mathbf{b}) \in \Lambda_{AB}$  and  $d\lambda$  is a shorthand notation for the measure on  $\Lambda_1 \times \Lambda_2$  for which the integral makes sense.

One of the most recent reformulations of CHSH theorem was given by Larsson and Gill [36]. Instead of  $\Lambda_{AB}$  they use a unique probability space  $\Lambda$  as in formula (6) saying :”...the particles travelling from the source carry some information about what the result would be of each possible measurement at the detectors. This information is denoted  $\lambda$  above, and can consist of anything , from a simple ”absolute” polarization to some complicated recipe of what each result measurement will be, for each setting of the detector parameter. What it is exactly is not important the very existence of such information will be referred to as ” Realism”, the  $\lambda$ , in a sense is ” the element of reality” that determines the measurement result.”. Without writing explicitly the formula (6) they conclude that under these prerequisites the CHSH theorem cannot be violated and therefore the Local Realistic hidden variable models are impossible. Next authors give an interesting discussion of how various experimental factors such as visibility , efficiency etc. may prevent the violation or prevent the application of the theorem getting a formula (9) which we rewrite here in a simplified form:

$$|E(\mathbf{a}, \mathbf{b}|\Lambda(A, B)) - E(\mathbf{a}, \mathbf{b}'|\Lambda(A, B'))| + |E(\mathbf{a}', \mathbf{b}'|\Lambda(A', B')) + E(\mathbf{a}', \mathbf{b}|\Lambda(A', B))| \leq 4 - 2\delta \quad ((9))$$

where  $\Lambda(C, D) \subset \Lambda$  denotes a subset of  $\Lambda$  describing the experiment  $(C, D)$  and  $\delta$  is some minimum probability of the overlap of the subsets used in the formula (9). In spite of the fact that the authors did not list very strong assumption concerning the existence of the joint distributions on the unique probability space  $\Lambda$  in their proof of (7) and (8) we completely agree with them that if the ”Realism ” is understood as a strict predetermination of the experimental outcomes at the source then Local Realistic hidden variable models are impossible. All local models able to reproduce the QT predictions [21,41,25,28,29] and to violate the equalities (7) are contextual as well as contextual are all SPCE. The correct formula is (8) and there is no overlap between  $\Lambda_{AB}$  and  $\Lambda_{CD}$  if  $C \neq A$  or  $B \neq D$ . Therefore BI or CHSH cannot be proven. The only formula which can be proven is the formula (8) with  $\delta = 0$  which of course does not violate QT predictions. Using the language of Larsson and Gill the information  $\lambda$  does not determine the future measurement results. The hidden information in the moment of the measurement ,carried by the particles, about the preparation at the source is stored in  $(\lambda_1, \lambda_2)$ . The information which predetermines the outcome of the measurement for the analyzer A is stored in  $(\lambda_1, \mathbf{a})$  where  $\mathbf{a}$  describes a microscopic state of analyzer A in the moment of measurement. The information what the outcome would be is not created at the source and decoded with

mistakes by the analyzer but it is created in the interaction with the analyzer and known only after the measurement is completed.

The hidden variable model of underlying sub-phenomena given by (8) is intuitive, local and contextual . According to Accardi's terminology [25] such probabilistic model describes an adaptive dynamical system. Another simple hidden variable model of SPCE has been proposed recently by Matzov [29].

We see that testing of BI-CHSH cannot help us to check the completeness of QT. We should test instead the predictable completeness of QT.

## 0.6 PURITY TESTS AND PREDICTABLE COMPLETENESS

Let us consider an experiment in which we have a stable source producing a beam of " identical invisible particles" whose intensity is measured by the clicks on some detector. When we pass this beam by some quantum filter F we obtain a beam having different properties and reduced intensity. The detailed discussion why quantum filters are not the selectors of preexisting properties is given in [42]. If by repeating our experiment several times we discover that the relative frequencies converge to some number  $p(F)$  we may interpret it as a probability that an individual particle from the beam will pass the filter F. According to SCI the claim that QT gives the complete description of the individual system being a member of some pure quantum ensemble may be only understood in the sense that the probabilistic predictions of QT provide complete description of the ensembles of the outcomes of all possible measurements performed on this pure quantum ensemble.

A standard interpretation of QT did recognize the importance of a pure quantum state and defined it as a state of physical system which passed by a maximal filter or on which a complete set of commuting observables was measured. The immediate question was what to do if we did not have a maximal filter or how could we know that a filter used was a maximal one? We found this definition highly unsatisfactory and we analyzed in 1973 various general experimental set-ups containing the sources of some hypothetical particle beams, detectors (counters), filters, transmitters and instruments [42]. This analysis led us to the various conclusions which are pertinent to the topic of this paper:

1) Properties of the beams depend on the properties of the devices and vice-versa and are defined only in terms of the observed interactions between them. For example a beam  $b$  is characterized by the statistical distribution of outcomes obtained by passing several replicas of this beam by all available devices  $d_i$ . A device  $d$  is defined by the statistical distribution of the results it produces for all available beams  $b_i$ . All observables are contextual and physical phenomena observed depend on the richness of the beams and of the devices.

2) In different runs of the experiments we observe the beams  $b_k$  each characterized by its empirical probability distribution. Only if an ensemble  $\beta$  of all these beams is a pure ensemble of pure beams we can associate estimated

probability distributions of the results with the beams  $b \in \beta$  and eventually with the individual particles who are forming these beams.

3) A pure ensemble  $\beta$  of pure beams  $b$  is characterized by such probability distributions  $s(r)$  which remain approximately unchanged:

(i) for the new ensembles  $\beta_i$  obtained from the ensemble  $\beta$  by the application of the  $i$ -th intensity reduction procedure on each beam  $b \in \beta$

(ii) for all rich sub-ensembles of  $\beta$  chosen in a random way

In order to test the validity of the Optical Theorem we decided to test whether the initial two-hadron states prepared for the high energy collision are mixed with respect to the impact parameter. Therefore we reviewed in a series of papers several non-parametric statistical purity tests which could be used [17] and together with Gajewski [51] we performed the purity tests for  $\pi^-d$  charge multiplicity distributions using the raw data from the Cambridge-Cracow-Warsaw collaboration in which the deuterium filled bubble chamber was exposed to a  $\pi^-$  beam of momentum 21GeV/c. We wanted to find significant differences between the data obtained in the different accelerator runs. If the initial state is pure, the different channels should be randomly distributed in time with some fixed probabilities of the appearance. If one concentrates on the appearance of two groups of the channels one can obtain the time ordered sequences of 0 and 1 such as : 1000110000... The randomness of these sequences can be tested in different ways . In one of the tests the hypothesis that the distribution was random could be rejected on the significance level: as low as of 0,0014. Since we considered our paper mainly as the illustration of various testing methods we did not insist on the importance of this result hoping that it will be confirmed by others.

In 1984 we noticed that the purity tests could be used also to test completeness of QT because :” The main feature of any theory with supplementary parameters is that the quantum pure ensembles become mixed ensembles of the individual systems characterized by the different values of these new parameters. There is a principal difference between a pure statistical ensemble and a mixed one. The pure ensemble is homogeneous , a mixed one should reveal a fine structure.” [16]. If the source is producing a pure beam of particles all runs of the experiment should be highly compatible. If the source is producing a mixed ensemble, the mixture could vary slightly from one run to another. We could also hope to change its composition by using some intensity reduction procedures. The purity test may be defined more rigorously as follows.

Let  $O$  be a stable source of particles and  $\gamma$  a measuring device of some physical observable  $\gamma X$ . A set  $S = \{x_k; k = 1, \dots, m\}$ , where  $x_k$  denote the measured values of  $\gamma X$ , is a sample drawn from some statistical population of the random variable  $X$  associated with the observable  $\gamma X$ . If  $b_i$  is a beam of  $m_i$  particles produced by the source  $O$  in the time interval  $[t_i, t_i + \Delta t]$  we obtain a sample  $S_i$  when  $\gamma X$  is measured the beam  $b_i$ . By using  $j$ -th beam intensity reduction procedure applied to the beam  $b_i$  we obtain a family of new beam  $b_i(j)$ ,  $j=1, \dots, n$ . Measuring  $\gamma X$  on the beams  $b_i(j)$  we obtain  $n$  new samples  $S_i(j)$ . We state that the beams produced by the source  $O$  are pure only if we cannot reject the hypothesis  $H_0$  :

$H_0$ : All the samples  $S_i$  and  $S_i(j)$  for different values of  $t_i$  and  $\Delta t$  are drawn from the same unknown statistical population.

To test  $H_0$  one has to use the statistical non-parametric compatibility tests such as : Wilcoxon-Mann-Whitney test, normal scores test, rank or run tests [17]. These tests can be used to analyze any existing experimental data in particular the data from SPCE. Of course the rejection of  $H_0$  proves only the impurity of the ensembles which were incorrectly described in QT as pure ensembles but it could be also an indication that QT is not predictably complete. It is not easy to show that QT is not predictably complete because the mathematical language it uses is very rich and flexible [27,42] allowing a good fit to the experimental data.

To prove the predictable incompleteness of QT we need something more. The results of any experiment may be represented as a time series of various possible outcomes. If there are  $k$  different outcomes possible QT describes this time series as a sample drawn at random from some particular multinomial probability distribution . The outcomes should appear therefore randomly in time with given probabilities. If one could detect some temporal fine structure in this time series or to find a stochastic model able to explain it then it would mean that QT does not provide a complete description of the experimental data obtained in this experiment. Several methods are used to study and to compare empirical time-series: frequency or harmonic analysis, periodograms etc.[52,58].

Due to the limited efficiencies of detectors and other imperfections of the experimental set-ups one has always a dilemma whether and how one should correct the data in order to obtain a "fair sample" of outcomes to be compared with the theoretical model of the phenomena. Adenier and Khrennikov analyzed recently the data from SPCE of Greg Weihs et al.[13] and found several interesting anomalies which survived various data correction attempts and which were not accounted for by the current description provided by QT. To elucidate these anomalies one would like to have more information about the calibration tests performed by the group before the experiment such as numbers of counts on the detectors: without spin analyzers on one of the sides and on both sides, with coincidence circuitry on and off, with the singlet source replaced by another source producing two beams having known spin polarization etc.

## 0.7 CONCLUSIONS

In spite of experimental imperfections we do not believe that the violation of BI in SPCE is due to the unfair sampling. The main reason, is that the probabilistic models used to prove BI are not valid for SPCE .

We hope that the arguments presented in this paper will cut short all speculations about the nonlocality observed in SPCE and will promote the statistical contextual interpretation (SCI) of QT .

SCI is free of paradoxes and shares Einstein's conviction that the probabilistic description of the phenomena is due to the lack of knowledge and control of the underlying sub-phenomena. At the same time SCI agrees with Bohr that the measured value of the physical observable is not predetermined and that it has

only meaning in a specific experimental context which must be always included in any model aiming to describe the sub-phenomena. SCI agrees with Bohr that QT gives only the probabilistic predictions for the phenomena but SCI does not say that more detailed description of the underlying sub-phenomena is impossible.

According to SCI the mysterious long range correlations in SPCE are due to the memory of the preparation at the source preserved in the sub-phenomena. Several local descriptions of sub-phenomena based on this idea e.g. [2,3,41,28,29] were able to reproduce the prediction of QT. Similar long range correlations exist in the macroscopic world. For example the violent earthquake in the middle of the ocean causes strong correlations between random variables such as :the force and the height of the Tsunami waves, force of the winds, number of victims etc. on far away shores [22]. Also in statistical physics there exist the long range correlations between coordinates and coarse -grained velocities of the Brownian particles which interacted in the past what was proven by Allahverdyan, Khrennikov and Nieuwenhuizen [53]. In view of this any speculation how the measurement performed on one photon influences the behavior of the other photon from the EPR pair is completely unfounded.

A general belief was that the language of Hilbert spaces and probability amplitudes used by QT could not be deduced from the classical theory of probability. This belief was shown to be incorrect by Khrennikov [54] who developed a probability calculus of conditional probabilities depending explicitly on experimental contexts and was able to reconstruct as a special case the probabilistic formalism of nonrelativistic quantum mechanics. In another paper [55] he showed that quantum averages  $\langle O \rangle = \text{Tr}(\hat{O} \rho)$  can be obtained as approximations of expectation values in some prequantum classical statistical field theory.

One cannot obtain a proof of incompleteness of QT by constructing ad hoc local hidden variable models able to reproduce some predictions of QT . The convincing proof of incompleteness would require a construction of a general model providing a consistent description of all phenomena described by QT and able to give more detailed predictions of these phenomena than those given by QT what seems to be a formidable task.

Testing the predictable completeness of QT seems easier and more promising. The outcomes of the physical experiments can be represented by some numerical time series. QT takes for granted the randomness of these time series and gives only the probabilities of the appearance of various outcomes. Any significant deviation from the randomness or the discovery of some reproducible fine structure in these series, not explained by QT, would prove that QT is not predictably complete and that a more detailed description of the phenomena is needed.

For example to describe effectively the behavior of cold trapped ions the continuous quantum evolution had to be supplemented by some quantum jumps obeying some stochastic Lévy process what was demonstrated by Claude Cohen-Tannoudji and collaborators [56]. Similarly it seems plausible that some new description may be needed in order to describe in detail the data from beautiful ex-



periments with ultra slow propagation of coherent light pulses in Bose-Einstein condensates reported recently by Lene Vestergaard Hau and collaborators [57].

The discovery of new fine structures in the data would be important by itself but also it would give additional clear argument against treating the quantum state vectors as attributes of individual physical systems which can be manipulated instantaneously. Such interpretation and instantaneous manipulations of qubits are often used in the domain of quantum computing [59].

Bohr considered QT as a theory of quantum phenomena and insisted on the "wholeness" of such phenomena [40]. If we have to talk about invisible sub-phenomena, what is inevitable, we have to use the most precise language in order to avoid paradoxes and confusion.

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